TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 568

CONTRIBUTION TO THE THEORY OF PROPELLER VIBRATIONS

By F. Liebers

From Zeitschrift für technische Physik
Volume X, 1929

Washington
June, 1930

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM NO. 568.

CONTRIBUTION TO THE THEORY OF PROPELLER VIBRATIONS.*

By F. Liebers.

Summary

Calculation of the torsional frequencies of revolving bars with allowance for the air forces. Calculation of the flexural or bending frequencies of revolving straight and tapered bars in terms of the angular velocity of revolution. Calculation on the basis of Rayleigh's principle of variation. Error estimation and the accuracy of the results. Application of the theory to screw propellers for airplanes and the liability of propellers to damage through vibrations due to lack of uniform loading.

Introduction

The construction of a propeller for static stresses by air forces and centrifugal forces offers no special difficulties. Nevertheless, propeller failures continue to occur, generally with very unfortunate results. Especially since in recent time, wood propellers are being increasingly replaced by light-metal propellers, there have been many accidents ascribable to failures resulting from the vibration of the propeller blades.

The possibility of the production and continuance of propeller vibrations is readily understood. The propeller revolves

*"Zur Theorie der Luftschraubenschwingungen." From Zeitschrift für technische Physik, Vol. X, 1929, pp. 361-369.

near the airplane and consequently in a region of disturbed air flow. A propeller behind the wing is especially disturbed by the vortex trail of the wing. On multi-engine airplanes, it sometimes happens that the different propeller circles overlap one another. Such influences alter the flow velocity and the angle of attack of the propeller blades, when they enter the turbulent region. Consequently, the propeller loading varies at different points. These disturbances, which generally depend on the revolution speed, cause propeller vibrations.

We do not yet have any accurate knowledge of the phenomenon of propeller vibrations. In the most common case, there are both torsional and flexural vibrations. We have no way of determining, however, which is the more dangerous. In order to obtain an insight mathematically, we will first inquire as to where the natural frequencies of the propeller blades lie for torsional and flexural vibrations, and which kind of vibrations can produce resonance as a result of disturbances due to the revolution speed.

By disregarding one of the kinds of vibration, we arrive at somewhat too large frequencies. From the standpoint of vibration resistivity, that form of propeller is therefore the best for which each kind of vibrations occurs independently of the other. The difference between the natural frequencies calculated here and the frequencies of the combined vibrations cannot be very great, however, since (as will be shown) the torsional and flexural frequencies differ greatly.

Torsional Vibrations

Preliminary Assumptions

Since we are not here considering a theory of the torsional vibrations of a propeller, but a mathematical estimation as to whether a normal propeller can be endangered by torsional vibrations, it is only necessary to calculate on the basis of very simple assumptions.

For this purpose, the actual propeller blade, which is tapered and warped from root to tip, is replaced by a blade of uniform cross section and torsional moment of inertia. Tapered blades will then have higher frequencies than the ones thus obtained.

Statement of Problem

If the air forces are disregarded, the revolution of the blade does not affect the torsion, since the centrifugal force yields no component at right angles to the blade. Hence we have the equation

$$J \frac{\partial^2 \Delta \alpha}{\partial t^2} = G J_q \frac{\partial^2 \Delta \alpha}{\partial x^2}$$

in which $\Delta \alpha$ is the angle of torsion with respect to the position of equilibrium; x, the length coordinate in the direction of the propeller radius; J, the inertia moment of a blade element of unit length about the axis of rotation; J_q, the cross-sectional moment against torsion; G, the modulus of shear; and t, the time.

To this equation there belongs another term which takes the air forces into consideration:

$$\left.\begin{aligned} J \frac{\partial^2 \Delta\alpha}{\partial t^2} &= G\, J_q \frac{\partial^2 \Delta\alpha}{\partial x^2} \\ &\quad - \frac{\rho}{2} (\omega^2 x^2 + v^2)\, b^2 \frac{\partial c_m}{\partial \alpha} \Delta \alpha \end{aligned}\right\} \qquad (1)$$

The second term represents the additional moment about the axis of rotation, which is produced by the air forces, when the angle of attack α is changed by the torsional vibration. c_m is the nondimensional moment coefficient of the air force about the axis of rotation. It can be derived from wind-tunnel tests. ρ is the air density and $\sqrt{\omega^2 x^2 + v^2}$ the resultant velocity with the components ωx = tangential velocity and v = flight velocity. b is the blade width.

The statement of the problem assumes the vibrations to be slow. Both the rotational velocity $\partial \Delta \alpha / \partial t$ and the related alteration of the effective angle of attack of a blade element are disregarded, as also the effect of the vortices periodically released by the vibrating blade. The insignificance of the resulting error is shown by the consideration of the reduced frequency,* which is a criterion for the slowness of a blade vibration. The reduced frequency of a propeller blade is found

*Birnbaum, "Das ebene Problem des schlagenden Flügels," Zeitschrift für angewandte Mathematik und Mechanik, 1924.

to be from 10 to 100 times smaller than that of a normal airplane wing.

Solution of the Torsional Equation

The lowest natural frequency of the vibration represented by equation (1) is obtained by the formula

$$\Delta \alpha (x,t) = y (x) \sin \lambda t.$$

Hence equation (1) becomes:

$$J G_q \frac{d^2 y}{d x^2} + \left[J \lambda^2 - \frac{\rho}{2} (\omega^2 x^2 + v^2) b^2 \frac{\partial c_m}{\partial \alpha} \right] y = 0 \qquad (2)$$

Equation (2) can be integrated in the form of an infinite converging series.* The solution of the latter, according to the frequency λ, reads:

$$\lambda = \sqrt{\frac{G J_q}{J l^2} \left(\frac{\pi}{2} + \Delta\right)^2 - \frac{\frac{\rho}{2} v^2 b^2 \frac{\partial c_m}{\partial \alpha}}{J}} \qquad (3)$$

Thereby, with a very good degree of accuracy,

$$\Delta = \frac{1 - 0.086 k - \sqrt{1 - 0.172 k + 0.0148 k^2}}{0.0216}$$

k being a nondimensional constant:

$$k = \frac{\rho \frac{\partial c_m}{\partial \alpha} \omega^2 b^2 l^4}{2 G J_q}.$$

*As Professor Reissner informs me, the effect of the air forces can be estimated (even without solving equation (2)) by assuming that $\partial c_m/\partial \alpha$ increases quadratically from the tip to the root of the blade. Then equation (2) becomes an ordinary vibration equation with constant coefficients. In this connection, however, the flight speed v must also be neglected.

Δ is zero when the revolution speed is $\omega = 0$. The second term under the radical disappears when the flight speed is $v = 0$. In this case equation (3) assumes the known value

$$\lambda = \frac{\pi}{2\,l}\sqrt{\frac{G\,J_q}{J}}. \qquad (4)$$

Results

If we calculate, according to equation (3), the torsional frequency for thin-bladed metal propellers at ordinary revolution speeds (1000-1500 r.p.m.) and flight speeds, we obtain values of the order of magnitude $\lambda = 6000\text{–}7000 \text{ min}^{-1}$. On the other hand, equation (4) yields frequencies only about 1.5% higher.

We thus obtain the remarkable result that the air forces have no appreciable effect on the vibration frequency of a propeller. This result is particularly impressive for an aeronautic engineer who has been accustomed to give quite a different weight to the air forces in comparison with the elastic and inertia forces.

A second result is furnished by the calculation that the torsional frequencies of ordinary propellers differ so widely from the frequency of the impulses connected with the revolution speed, that resonance in the form of torsional vibrations does not enter into the question.

Bending Vibrations

Preliminary Assumptions

After finding that the torsional vibrations can hardly endanger the propeller, the next step is to investigate the flexural vibrations. In calculating the flexural or bending frequency of a revolving propeller blade, it is to be noted that, due to the centrifugal force, the curve of the bending moments is a function of the angular velocity of the propeller and also that the bending frequency depends on the revolution speed. It seems practically impossible to determine the line of elastic vibration by solving the commonly employed integral-differential equation for the bending vibrations of a revolving propeller. Therefore the bending curve is here sought by the use of the variation principle of Rayleigh ("Theory of Sound," Vol. I, Sections 88-89), according to which, among all the possible bending curves, that one is used for which the frequency is the lowest.

This calculation requires ideal preliminary assumptions. Hence the propeller blade is again regarded as a simple beam or bar. The bent and twisted shape of the blade is therefore disregarded. In order, however, to come nearer the reality in one important point, the problem is treated both for uniform and variable cross sections and also for a variable cross-sectional inertia moment. Thereby the cross section and the inertia moment according to the power curves are assumed to vary with the

radius. Cross section: $$\left.\begin{array}{ll} F = F_0\ (1 - \xi)^{\kappa} & \xi = \frac{x}{l} \\ J = J_0\ (1 - \xi)^{\vartheta} & 0 \leq \xi \leq 1 \end{array}\right\} \qquad (5)$$
Inertia moment:

x being the distance of a cross section from the middle point and l the blade length. The exponents κ and ϑ are variable parameters, to be adapted to the given propeller. (According to the measurements of ordinary propellers $\kappa \simeq 1$ and $\vartheta \approx 2 - 2.5.$)

Furthermore, it is assumed that the bending vibrations are perpendicular to the plane of revolution of the propeller blades. The more common case, in which they are at an angle to the plane of revolution, can be similarly treated without difficulty. Lastly, the effect of the air forces can be disregarded, which is justified by a mathematical calculation, the same as for the torsional vibrations.

The Method

The calculation of the bending frequency of the revolving propeller blade, on the basis of the Rayleigh principle, can be accurately made by the method of the calculus of variations. In the present instance only the so-called "direct" method is applicable. The initially unknown line of elastic vibration takes the form of an infinite series $y\ (\xi) = \Sigma\ c_\nu\ f_\nu\ (\xi)$, whereby the functions f_ν form a completely fixed orthogonal system, so that, through the theorem, any bending line can be determined to the desired accuracy.

This method is troublesome, however, and does not always yield a satisfactory solution. We can simplify the present variation problem considerably by transforming it into an ordinary minimum problem. This can be accomplished by employing only a certain, but physically grounded group of simple infinitely many bending curves in concurrence for the given minimum problem. For this purpose a group of curves was selected as follows.

In the case of a beam of uniform cross section fixed at one end and without centrifugal forces, the line of elastic vibration practically coincides with the static bending from its own weight (the frequency difference, calculated on the assumption of one or the other curve, being less than 0.5%). We now first develop, from the mathematically simple static bending curve, a group of curves by their constant deformation in the direction of the stretching effect of the centrifugal force which increases with the revolution speed. Moreover, since tapered propeller blades are to be included in the scope of the calculation, we must also see that the basic group of elastic curves includes those produced from the bending curves for cylindrical bars by constant deformation in the direction of an enlargement of the mean bend. In a group of bending curves selected from these viewpoints, the difference cannot be very great between the true elastic curve of the rotating cylinder or tapered bar and one of the infinitely many curves of the plotted group. It cannot be very great, especially when the four end conditions and the uniformity conditions of the problem are satisfied by all the curves of the group. (In general there is the advantage that Rayleigh's theorem expresses a minimum condition. In the vicinity of a minimum the deviations are naturally small.)

The following single-parameter group of functions for the lines of elastic vibration satisfies all the above requirements.

$$\frac{y\,(\xi,n)}{y_{max}} = \frac{1}{n+2}\left[(1-\xi)^{n+3} + (n+3)\ \xi - 1\right] \qquad (6)$$

n is the variable parameter. For $n = 1$, equation (6) gives the bending curve at the load produced by the weight of the blade. For $n < 1$, the curves 6 correspond to the bending curves for tapered blades at vanishing centrifugal forces. The four end conditions, that at the point of fixation $(\xi = 0)$ the amplitude $(y = 0)$ and the differential quotient $(y' = 0)$ vanish and at the free end $(\xi = 1)$ the bending moment $(y'' = 0)$ and the shearing force $(y''' = 0)$ all equal zero, are all satisfied by formula (6) for $n > 0$. The group of curves 6 is plotted in Figure 1. The strongly curved lines are available as elastic lines for small revolution speeds and centrifugal forces. With increasing revolution speed, the flatter curves apply, which become continually flatter, due to the centrifugal force, until they become straight at infinite revolution speed, where the elastic forces are negligible.

Calculation of the Bending Frequency

The frequency is determined by the Rayleigh method with the aid of the energy theorem according to which the kinetic energy must equal the sum of the deformation energy and centrifugal energy for the maximum amplitude of vibration. (The centrifugal energy is the product of the centrifugal force and the radial displacement of its point of application during a vibration.) The energy equation is

$$\left.\begin{aligned} \frac{\rho_m}{2}\, l\, \lambda^2 \int_0^1 F\, y^2\, d\,\xi = {} & \frac{E}{2\, l^3} \int_0^1 J\, y''^2\, d\,\xi \\ & + \frac{\rho_m}{2}\, l\, \omega^2 \int_0^1 \left\{ \int_0^\xi y'^2\, d\,\xi \right\} F\, \xi\, d\,\xi \end{aligned}\right\} \qquad (7)$$

(Stodola, "Dampf- und Gasturbinen," paragraph 195). ρ_m denotes the density of the material and E the modulus of elasticity. If the values for $F(\xi)$ and $J(\xi)$ (equation 5) and the indeterminate expression for the line of elastic vibration (equation 6) are introduced, then equation (7), solved according to the frequency λ, yields λ as a function of the parameter n. For given constructional quantities (l, F, J, E, ρ_m) and a given revolution speed ω of the propeller, we can therefore find the n for which λ becomes a minimum.

The solution of equation (7), according to the frequency

λ, can be written*

$$\lambda^2 = \frac{E\ J_o}{\rho_m\ F_o\ l^4}\ X_1\ (n,\kappa,\vartheta) + \omega^2\ X_2\ (n,\kappa). \tag{8}$$

For the case of zero rotation ($\omega = 0$), the second term drops out, and the bending frequency is

$$\lambda^2_{\omega=o} \equiv \lambda_o^2 = \frac{E\ J_o}{\rho_m\ F_o\ l^4}\ X_{1\,min}\ (n',\kappa,\vartheta) \tag{9}$$

It is intended to indicate by n' that the minimum of the function X_1 has already been found. $X_{1\,min}$ is a constant for certain values of κ and ϑ. Equation (8) can now be written

$$\lambda^2 = \frac{X_1\ (n,\kappa,\vartheta)}{X_{1\,min}\ (n',\kappa,\vartheta)}\ \lambda_o^2 + X_2(n,\kappa)\omega^2 \tag{10}$$

or

$$\left(\frac{\lambda}{\lambda_o}\right)^2 = \overline{X}_1\ (n,\kappa,\vartheta) + X_2\ (n,\kappa)\left(\frac{\omega}{\lambda_o}\right)^2 \tag{11}$$

The equation for λ is homogeneous in λ, λ_o and ω. Hence there are only two fundamental variables, λ/λ_o and ω/λ_o, i.e., for all propellers of like taper, a single curve is sufficient, namely λ/λ_o plotted against ω/λ_o. This is an important simplification. In Figure 2, λ/λ_o is plotted against ω/λ_o for various tapers κ, ϑ. Figure 2 was plotted with the aid of equation (11), by determining the parameter n for different values of ω/λ_o so as to make λ/λ_o a minimum.

*A detailed description of the method of calculation, accompanied by explicit formulas, will appear in one of the early numbers of "Luftfahrtforschung" as a report of the Deutsche Versuchsanstalt für Luftfahrt.

Hence we have in Figure 2, for every propeller of any strength, of any dimensions and of any taper (i.e., for any value of λ_0), a chart from which the bending frequency at any revolution speed can be determined in a few seconds. Of course the frequency λ_0 must be previously found from equation (9). This is done for the practically important cases of uniform taper ($\kappa = 1$) and between $\vartheta = 0$ and $\vartheta = 3$ of varying reduction of the cross-sectional inertia moment (Fig. 3). Figures 2 and 3 answer, once for all, the question regarding the bending frequency of any propeller in terms of the revolution speed. In cases where it seems necessary or advisable, λ_0 can be determined experimentally in a simple manner, instead of taking it from Figure 3.

The different curves of Figure 2, for different pairs of values of κ and ϑ, differ but little in the practical range of the κ and ϑ values. Noticeable discrepancies occur only at very great tapers. We find, therefore, that two propellers (or beams), differing in dimensions and material constants (within broad limits) and even in taper, but having like natural frequencies, have nearly the same bending frequencies even during rotation.

The relative position of the curves in Figure 2 is correct, showing that the frequency is increased by increasing the taper. For the same cross-sectional taper, the propeller with the greater reduction of the inertia moment has the smaller natural frequencies.

If it is desired to dispense with the chart, good approximation formulas can be developed for the plotted curves. The following is a good interpolation formula for the practical tapers. For

$$0 \leqq \frac{\omega}{\lambda_0} \leqq 3, \quad \frac{\lambda}{\lambda_0} = 1 + \frac{7\ \omega^2}{\lambda_0\ (6\ \lambda_0 + 7\ \omega)} .$$

For

$$\frac{\omega}{\lambda_0} \geqq 3, \quad \frac{\lambda}{\lambda_0} = \frac{1}{3} + \frac{\omega}{\lambda_0}.$$

Accuracy of the Calculation*

In support of the adopted method of calculation, we make the following estimate of errors.' The actual bending frequencies, according to Rayleigh's theorem on which our method is based, cannot be greater than those found here.

If we consider equation (11) for $(\lambda/\lambda_0)^2$ and assume that not the bending lines according to equation (6) but the true bending lines are taken as the basis of the calculation. Then there is no change in the representation of the frequency and equation (11) is absolutely correct. Equation (11) (now independent of any physical significance) reaches its absolute minimum, however, when all of the summands (terms in the summation) are as small as possible. Otherwise λ would not equal λ_0 for $\omega = 0$. The same is true of the function X_2. If $X_{2\,min}$

*This section goes into greater detail than the rest of the lecture, since it seems important as proof of the method of calculation.

were not equal to 1, then, in the absence of elasticity $(\lambda_o = 0)$, λ would not equal zero. It is obvious that such must be the case from the simple vibration calculation for the fiber under the influence of the centrifugal force alone (sling).*

The minimum value of equation (11) is therefore $1 + (\omega/\lambda_o)^2$. For the frequency $(\lambda/\lambda_o)^2$, however, this is only a lower limit (Schranke) which is never used*** because the minimum of a sum of functions is only equal to the sum of the minima of the individual summands, when the latter are obtained for the same values of the variables. Obviously that is not the case here, since the bending lines for $\lambda_o = 0$ and $\omega = 0$ are very different (different values of n in expression 6. We therefore have the inequality

$$\left.\begin{aligned} \frac{\lambda}{\lambda_o} &> \sqrt{1 + \left(\frac{\omega}{\lambda_o}\right)^2} \\ \text{or} \qquad \lambda^2 &> \lambda_o^2 + \omega^2 \end{aligned}\right\} \tag{12}$$

Since this relation was deduced independently of the quantities κ and ϑ, it applies quite generally to every type of beam.**

*Attention is called to the fact that the group of adopted bending lines (equation 6) is already so favorable, that even the minimum of our function $X_2(n,\kappa)$ has the value 1, as in the case of the assumption of the true elastic lines.

**In contrast with the lower limit (Grenze) which is actually reached.

***As I subsequently learned, this lower limit (Schranke) which was used for the present special case, was contained in a general theorem of Lamb and Southworth (Proc. Roy. Soc., Vol. 99, 1921), according to which the square of the frequency of an elastic system, subjected to several forces, which, independently of one another, affect the potential energy of the system, is always greater than the sum of the frequency squares which the system would have if only one of the forces acted on the system.

In Figure 2 the lower limit (Schranke) (equation 12), which holds for all propellers, is represented by a dash line.

Moreover, Figure 2 shows the greatest conceivable per cent error. In the most unfavorable case for propellers of the taper under consideration, it is about 5% and decreases toward zero both with vanishing and with increasing centrifugal force. As a matter of fact, the discrepancies should be considerably smaller than indicated, because the frequencies, calculated on the basis of the bending lines (equation 6), come decidedly nearer the reality than the values of the lower limit (Schranke), which has only a formal mathematical significance but no physical import. For rough calculations, however, the simple formula (12) can be used with good results. In using this formula, one would surely be on the safe side.

Up to this point, the error estimation has referred to the function λ/λ_0 in terms of ω/λ_0. It still remains to test the accuracy of the calculated values λ_0 and the value of $X_{1\,min}(n,\kappa,\vartheta)$ in equation (9). In the case of the cylindrical bar $(\kappa = \vartheta = 0)$ it is found that $X_{1\,min} = 12.36$, the exact value being

$$\lambda_0^2 = 12.359 \ldots \frac{E\ J_0}{\rho_m\ F_0\ l^4}$$

The agreement is therefore complete.

There are no accurate data available for the comparison of the static frequency (Standfrequenz) of tapered blades. Hence we resort to the approximate data of Hort* for our investigation. On the basis of Hort's deductions we obtain, for example,

$$\lambda_0^2 = 51.6 \frac{J_0 E}{\rho_m F_0 l^4}$$

for the case of simultaneous linear decrease of cross section and moment of inertia. In this case, according to our calculation (Fig. 3), $X_{1\,min} = 51.10$. The difference is less than 1%. In fact, this value, as the smaller according to Rayleigh's theorem, is the more nearly correct one. Likewise, according to Hort, the numerical factor in λ_0 is 43.75 for the case of linear taper of the cross section and of quadratic decrease in the inertia moment, while we obtain 40.75 (Fig. 3). In this case, the latter value is about 7% better. The difference is still more noticeable in case of greater tapers. The reason lies in the fact that the Hort formulas apply only to slightly tapered blades.**

On the basis of this comparison, the here-calculated bending frequencies are found to be useful in the absence of centrifugal forces and to surpass the results of previous calculations. If λ_0 is experimentally determined in unusual cases, the calculation of the frequencies for different angular veloci-

*W. Hort, "Berechnung der Eigentöne nicht gleichförmiger, insbesondere verjüngter Stäbe," Zeitschrift für technische Physik, Vol. VI, 1925, p. 181.

**The reason does not lie in the fact that, for example, in the case of quadratic reduction of the inertia moment, $J = J_0(1-\xi)^2$, the pertinent theorem of Hort reads $J = J_0(1-\xi-1/4\,\sin\pi\,\xi)$, since the two theorems can hardly be distinguished numerically.

ties is made according to Figure 2 with the above-mentioned accuracy.

The error estimates indicate that the accuracy required in practice is generally excelled by the above method of calculation.

Comparison with Older Formulas

The above results regarding flexural vibrations are of general importance independently of the problem of propeller vibrations. The formulas have the advantage over older formulas of being more accurate and of being applicable throughout the whole range of angular velocities from $\omega = 0$ to $\omega = \infty$.

The known formulas for the bending frequency of rotating beams have the form

$$\lambda^2 = C_1 \frac{E\,J_0}{\rho_m^*\,F_0\,l^4} + C_2\,\omega^2 \qquad (13)$$

where C_1 and C_2 are constants. This formula represents a good approximation only within a comparatively small range of ω, because equation (13) is based on only a single form of vibration, which is more or less adapted to a very definite relation of elastic force to centrifugal force. For example, if the frequency according to equation (13) approaches the value $\lambda^2 = C_2\,\omega^2$ with increasing ω, then $\lambda = \omega$ for $\omega = \infty$.

The first application of Rayleigh's line of reasoning to a similar problem (bar at the edge of a revolving disk) was made to my knowledge by Stodola. The formula used by him cannot be transferred, however, to the case of an aircraft propeller, as it would give $\lambda < \omega$ for large ω in contradiction with equation (12). From the standpoint of the strictly formulated theorem of Rayleigh, it should be noted regarding Stodola's calculation that the proposed changes in form represent no possible bending-vibration curves, since the end conditions at the free end of the bar are not satisfied. This is expressed in Stodola's formula corresponding to our equation (8), in that the former, by consistent treatment, would yield a physically absurd result. This assumption seems very opportune because it follows (especially in still more intricate problems than the one under consideration) that Rayleigh's theorem should be employed in the strictest possible form, in order to avoid wrong conclusions.

A previous investigation, which yielded a result related to formula (13), was made by Southwell and Gough.* In their report, the true frequency is estimated by the formation of an upper and a lower limit. The latter was deduced from the above-mentioned theorem of Lamb and Southwell in agreement with the inequality 12. The upper limit, however, has the form 13, so that, with increasing ω, the distance between the upper and lower limits can be made as great as desired.

---

*British A.R.C. Reports and Memoranda No. 766: "The Free Transverse Vibration of Airscrew Blades, 1921-22.

Practical Results

We first find that the natural frequency for bending is always greater than the induced frequency ω. This is unfavorable, since it is only when the induced frequency is considerably above the natural frequency that (with a slight internal damping) we can calculate on a complete damping of the vibrations. It depends also on how closely the natural frequencies approach the revolution speeds for propellers of the usual dimensions.

In Figure 4 (curves I-IV) the bending frequencies are plotted against the angular velocity for an especially thin-bladed metal propeller on the basis of Figure 2. Curves I-IV correspond to four propellers of like root cross section but different tapers. I is for the cylindrical bar, while III and IV (linear cross-sectional, quadratic or cubical moment of inertia reduction) correspond to practical propeller forms.

In order to determine the danger limits for a given propeller blade, one must know the vibration resistivity of the blade, as expressed by the maximum permissible amplitude, and also the magnitude of the disturbing force. Both of these can be determined.* Then we can conclude, on the basis of the known reso-

*The impulse received by a propeller in the vicinity of an airplane wing (due to variation in the flow velocity and the angle of attack of the propeller blade) can be approximately calculated with the aid of simple formulas of the wing theory. For the turbulent field behind a wing, we already have at our disposal the results of flight tests made for another purpose. (M. Schrenk, "Ueber Profilwiderstandmessung im Fluge nach dem Impulsverfahren," Luftfahrtforschung, May 18, 1928. For translation, see N.A.C.A. Technical Memorandums Nos. 557 and 558: "Measurement of Profile Drag on an Airplane in Flight by the Momentum Method," Parts I and II, March, 1930.

nance curves, that the amplitudes, at a certain ratio of the induced frequency to the natural frequency, exceed the bounds established by the vibration resistivity. If it be provisionally assumed that this limit was reached for $\omega/\lambda = 0.8$, then (Fig. 4) the points of intersection of the straight line $\lambda = 1.25\ \omega$ with the various frequency curves indicate the revolution speeds at which the given propeller blade begins to develop dangerous vibrations. The cylindrical blade was already endangered at $\omega = 70\ \text{sec.}^{-1} \simeq 700$ r.p.m.; the tapered blade III first at about 1500 r.p.m.

On the other hand, the frequencies calculated for a considerably larger propeller (curve V) are plotted in Figure 4. The propeller has the dimensions of a Reed propeller, as used on three-engine commercial airplanes (Fig. 5). Many propellers of this type have been damaged in flight.

From the course of curve V, it is seen that, in the region of normal revolution speeds, the natural frequencies of the propeller are certainly not in the neighborhood of the revolution speed. Very likely, however, they assume values twice as great as the revolution speed. In fact, this pronounced resonance begins between 1000 and 1200 r.p.m.

If we consider the field of flow of a propeller of a three-engine airplane (Fig. 5), it is obvious that the propeller passes through two disturbed regions at each revolution. For the lateral propellers, both impulses occur in passing the wing at a distance of only 1/3 the wing chord. At the inner point, they also encounter the slipstream of the middle propeller, which is spread out by the effect of the fuselage. The middle propeller functions at about double the distance of the side propellers from the wing. It is only very slightly affected by the wing, as well as by the other propellers. In fact, central propellers have fewer disadvantages, though not entirely free from them.

On the whole, the above practical example shows a very good agreement between calculation and experience. The calculation shows further (Fig. 4) that, in one case, it would be better, under certain conditions, to use a more flexible propeller with such bending frequencies that, under normal conditions, the induced frequency would be greater than the natural frequency. The vibrations would then be well-damped.

Conclusions

The practical results of the investigation are:

1. From the standpoint of resistivity to vibrations, that propeller is the most favorable for which the vibrations for each degree of freedom can take place independently of the other.

2. The torsional vibrations of propellers are generally so high, that the danger of resonance from the vibrations, which equal or double the revolution speed, is eliminated.

3. The elastic and inertia forces exceed the air forces so much that the latter are practically negligible for the frequencies of the propeller vibrations.

4. Flexural vibrations seem to be the principal cause for propeller failures. The preliminary calculation of the bending frequencies of propeller blades in terms of the r.p.m. can be made more quickly and with greater accuracy by the method described.

5. Every propeller develops resonance vibrations at sufficiently high revolution speeds.

6. A decided cross-sectional taper is favorable for increasing the bending frequency, but it is better for the moment of inertia to be everywhere as great as possible.

7. The location of the propeller with respect to the wing is very important. Even very strong propellers may be endangered by developing resonance vibrations from impulses of double the r.p.m. due to unfavorable installation. In such cases, a more flexible propeller is sometimes preferable.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

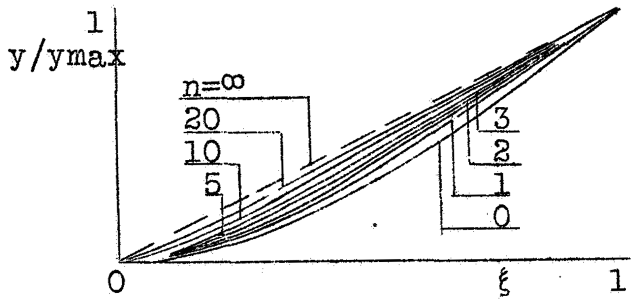

Fig.1 Lines of elastic vibration (eq.6)

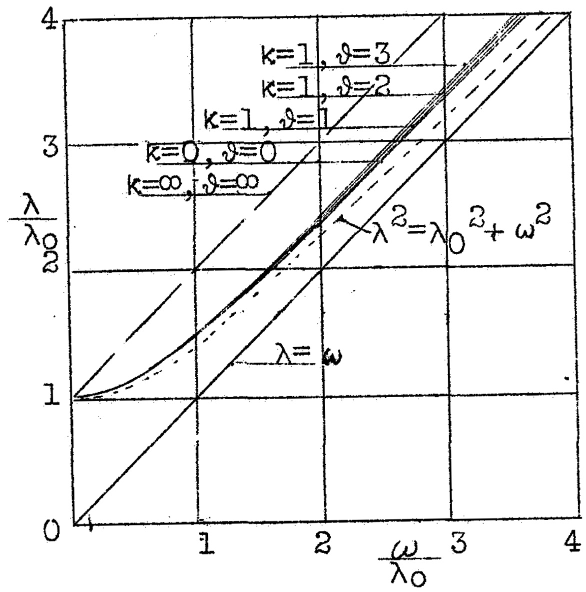

Fig.2 Nondimensional representation of the bending frequencies of revolving bars (propeller blades) of different tapers plotted against the angular velocity (eq.11)

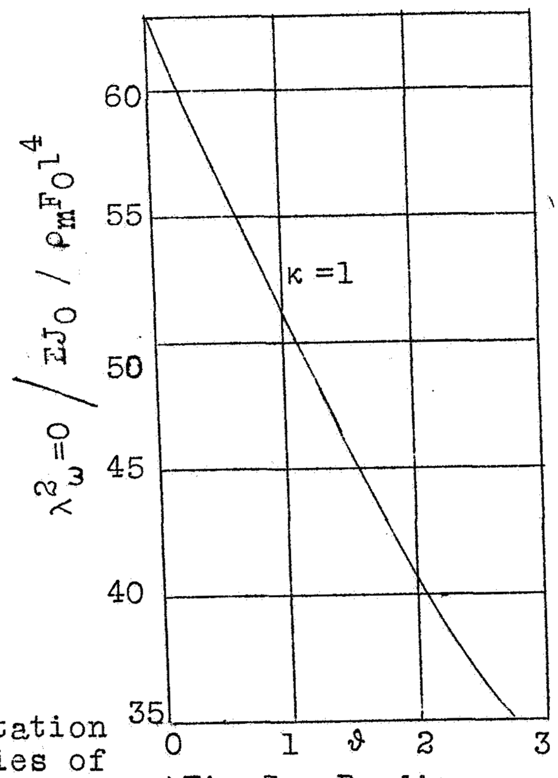

Fig.3 Bending frequencies of nonrevolving bars of uniform taper (κ =1) and varying reduction of the cross-sectional inertia moment (ϑ variable).

	E (kg cm^{-2})	ρ_m (kg^2 cm^{-4})	l (cm)	J_o (cm^4)	F_o (cm^2)	k	ϑ
I						0	0
II							1
III	7.8×10^5	3×10^{-5}	150	13	40	1	2
IV							3
V				60	85	1	2

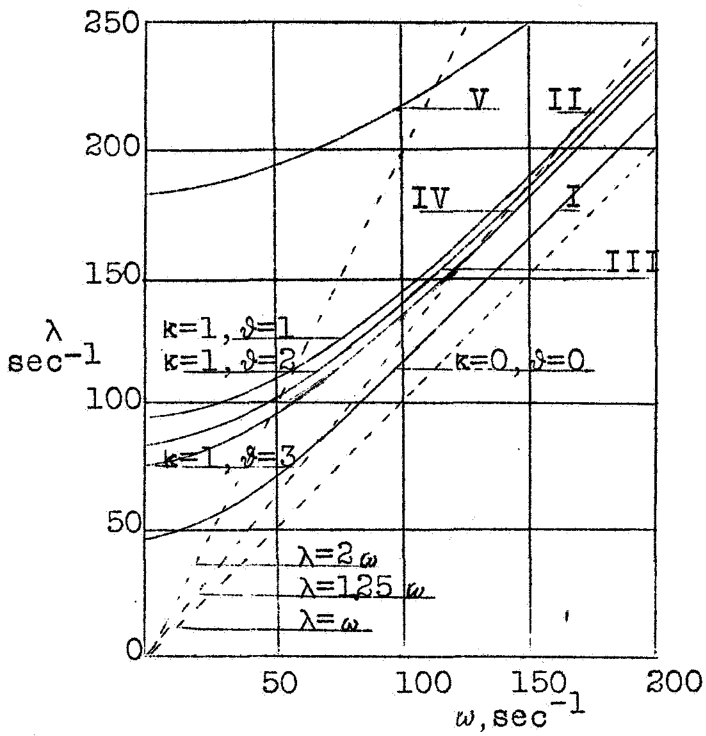

Fig.4 Bending frequencies of different propellers plotted against the angular velocity.

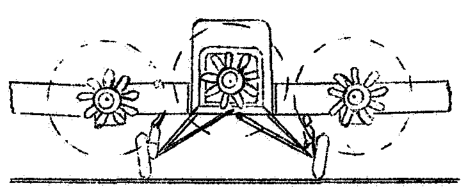

Fig.5 Arrangement of propellers on 3 engine airplane (from "Flugsport" 1927, p.489).